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A class of similar solutions of the non-linear diffusion equation

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Abstract. The existence of a general class of similar solutions of the diffusion equation is demonstrated, when the boundary conditions vary as a simple power of time, and the transport coefficient varies non-linearly as a power of the concentration. Although earlier workers have described specific exact solutions, which are members of this class, the class as a whole has not previously been investigated. These exact solutions, and one previously unreported, are described briefly. A simple method is given for the numerical integration of the characteristic differential equation of the profile for the case when an exact solution is not available.

The use of these solutions in studies of laser interaction with solid targets, and as test problems for thermal conduction routines, is briefly discussed.

1. Introduction

The dominant role of thermal conduction in the early stages of laser interaction with solid targets is widely appreciated (Babuel-Peyrissac *et al* 1969, Caruso and Gratton 1969, Saltzmann 1973, Pert 1974). Unfortunately, in these applications the thermal conductivity is non-linear being a strong function of the temperature of the conducting bodies (usually electrons). In general the inclusion of this non-linearity greatly restricts the number of analytical solutions which can be found. In this paper we present a general class of self-similar solutions to the problem of heat conduction by a medium with non-linear conductivity, which have relevance to the early stages of laser–plasma heating phenomena.

Similarity solutions for the diffusion equation have been widely investigated and some general methods (Ames 1965, 1972) are available. Many of these (Philip 1960) make use of the well known Boltzmann transformation (Crank 1975) in which the similarity is expressed in terms of the variable $x/\sqrt{t}$, where x is the spatial variable and t is time. The class of solutions examined in this paper is distinct from these and is, when applicable, simpler to use. Some of the simpler members of this class have already been described by Carslaw and Jaeger (1959), Pattle (1959), Boyer (1962), Ames (1966), Zel'dovich and Raizer (1967) and Anisimov (1972), but consideration of the class as a whole has not been presented, possibly due to the difficulties in performing the necessary numerical integration. In this paper we show how this difficulty may be simply overcome.

In this paper we develop the solutions for the particular case of the transfer of heat in an isotropic medium. However, the method is, of course, more general, and is applicable to any problem of diffusion: particle diffusion (Tuck 1976), radiative diffusion (Zel'dovich and Raizer 1967), plasma wave diffusion (Lonngren *et al* 1974) or transmission line theory (Boyer 1962).

We shall consider the time-dependent propagation of heat in uniform media, under conditions of one-dimensional symmetry, with boundary conditions which vary as a simple power of time. Provided the initial temperature of the medium is very small we shall show that these problems can be cast into a self-similar form, and shall demonstrate how the exact solution may be obtained.

The conduction of heat in a uniform medium, whose properties vary non-linearly with temperature, is most simply studied by writing the equation of heat transport in the form:

$$\frac{\partial \epsilon}{\partial t} = \operatorname{div} \boldsymbol{q} \qquad \boldsymbol{q} = -\kappa \operatorname{grad} \epsilon \tag{1}$$

where κ is a thermal conduction coefficient, and ϵ, the specific internal energy, is a function of temperature alone, since the medium is uniform in density. We shall assume that the thermal conductivity varies as a simple power of the internal energy (Zel'dovich and Raizer 1967):

$$\kappa = a\epsilon^n. \tag{2}$$

2. The existence of self-similar solutions

Since we have reduced the problem to only one spatial variable, say x, there are two independent variables only, namely time t and x, and one dependent variable ϵ. The problem is characterized by the parameter a from equation (2) and the initial and boundary values. In general we should specify an initial energy ϵ_0 for the medium, however, if the initial temperature is very small we may with little error take $\epsilon_0 = 0$. The boundary values must be characterized on the boundary surface, and may, as is usual with diffusion problems, specify either the parameter or the flux at the surface, in the form $S(t) = S_0 f(t/\tau)$, containing two parameters S_0 and τ. In the special case that $S(t)$ is a simple power of time:

$$S(t) = S_0 t^m \tag{3}$$

there is only one characteristic parameter S_0.

We may now use dimensional analysis to investigate the functional form of $\epsilon(x, t)$. In general we have five variables, ϵ, x, t, a and S_0. The rank of the dimensional matrix is three; therefore the number of dimensionless products in the complete set is two (Langhaar 1951). Hence by Buckingham's theorem

$$\epsilon(x, t) = h(t) f(x/g(t)) \tag{4}$$

and thus the problem is cast into a self-similar form with variable

$$\xi = x/g(t).$$

Dimensional analysis will also enable us to find the functional forms of $g(t)$ and $h(t)$ for particular cases. In order to illustrate the calculation of $f(\xi)$ we shall consider in detail only the planar one-dimensional problem, where $S(t)$ is the energy flux incident on the surface $x = 0$ of a medium occupying the half-space $x > 0$. In this case

$$\begin{aligned} h(t) &\propto \left(\frac{S_0^2 t^{(2m+1)}}{a}\right)^{1/(n+2)} \\ g(t) &\propto (a S_0^n t^{(nm+n+1)})^{1/(n+2)}. \end{aligned} \tag{5}$$

3. Separation of the differential equation

The existence of this self-similar solution implies that the differential equation (1) is separable, indeed through equation (5) we could specify the similarity variables. However, it is more convenient for our purpose to consider the separation directly: thus we put

$$\epsilon = h(t)f(\xi) \qquad \xi = x/g(t). \tag{6}$$

It is readily shown that a sufficient condition that the solution be separable is:

$$\dot{h}(t) = \frac{\alpha h(t)}{g(t)}\dot{g}(t) \qquad h(t) = \beta(g(t))^{\alpha} \tag{7}$$

where α and β are constants. The values of α and β are found by matching the solution to the boundary condition in either of the forms:

$$-a\epsilon^{n}\frac{\partial\epsilon}{\partial x}\bigg|_{x=0} = S_0 t^{m}$$

or

$$\int_0^{\infty} \epsilon \, \mathrm{d}x = \int_0^{t} S_0 t^{m} \, \mathrm{d}t = \frac{S_0 t^{m+1}}{m+1}. \tag{8}$$

In general it is simpler to use the first form, although the second allows the inclusion of instantaneous heating within this formalism, as we show later. Hence:

$$\alpha = \frac{2m+1}{nm+n+1} \qquad \beta = \left[\left(\frac{2m+1}{A(n+2)}\right)^{m}\frac{S_0}{aB}\right]^{1/(nm+n+1)} \tag{9}$$

where A is the separation constant and

$$f^{n}\frac{\mathrm{d}f}{\mathrm{d}\xi}\bigg|_{\xi=0} = -B \qquad \int f \, \mathrm{d}\xi = \frac{2m+1}{(n+2)(m+1)}BC. \tag{10}$$

Since A has the dimensions of a it is convenient to introduce the dimensionless form $C = a/A$. Hence we find:

$$\xi = x\left[\left(\frac{(n+2)}{(2m+1)C}\right)^{n+1} a\left(\frac{S_0}{B}\right)^{n} t^{nm+n+1}\right]^{-1/(n+2)} \tag{11}$$

and

$$h(t) = \left[\frac{n+2}{(2m+1)aC}\left(\frac{S_0}{B}\right)^{2} t^{2m+1}\right]^{1/(n+2)} \tag{12}$$

in agreement with the dimensional analysis (5).

The constants B and C are arbitrary, and it is easy to show that x and ϵ are invariants under transformations of B and C. In common with previous workers (Zel'dovich and Raizer 1967) we may put $B = 1$ and $C = (n+2)/(2m+1)$ but for the present prefer to leave both B and C arbitrary.

4. The characteristic profile

The characteristic thermal profile is determined by the function $f(\xi)$ which satisfies the differential equation obtained by separating (1):

$$C\frac{\mathrm{d}}{\mathrm{d}\xi}\left(f^n\frac{\mathrm{d}f}{\mathrm{d}\xi}\right)=f-\frac{nm+n+1}{2m+1}\xi\frac{\mathrm{d}f}{\mathrm{d}\xi} \tag{13}$$

subject to the boundary conditions given by (10)† and $f\to 0$ as $\xi\to\infty$. This latter boundary condition presents some problems. If we examine the form of (13) near $f=0$ we find that:

$$\lim_{f\to 0} f^{n-1}\frac{\mathrm{d}f}{\mathrm{d}\xi}=-\frac{nm+n+1}{(2m+1)C}\xi_0 \tag{14}$$

where $f(\xi_0)=0$. The solution of f satisfying (13) and the boundary conditions is thus given by the integral of (13) for $\xi\leqslant\xi_0$, and zero thereafter. This behaviour with a finite thermal front at $\xi=\xi_0$ is characteristic of non-linear thermal conduction problems (Zel'dovich and Raizer 1967).

Equation (14) has a simple physical interpretation. Near the thermal front, thermal conduction predominately supplies heat to maintain the motion of the front, so that:

$$\frac{\partial\epsilon}{\partial t}\simeq -v\frac{\partial\epsilon}{\partial x}=\frac{\partial}{\partial x}\left(a\epsilon^n\frac{\partial\epsilon}{\partial x}\right) \tag{15}$$

where v is the velocity of the front:

$$v=\left(\frac{\mathrm{d}x}{\mathrm{d}t}\right)_{\xi_0}\simeq -a\epsilon^{n-1}\frac{\partial\epsilon}{\partial x}\bigg|_{\xi_0}. \tag{16}$$

Equation (14) is obtained directly since $g(t)\dot{g}(t)/h^n(t)=(nm+n+1)a/(2m+1)C$.

If $n>1$, all solutions which cut the ξ axis do so normally. Since f must be zero at some point, the required solution is the one which satisfies both (10) and (14). Alternatively, if we recall that B is arbitrary and simply a scale factor, we may regard equation (14) as the boundary condition for some fixed value of ξ_0 (say $\xi_0=1$) and regard B as determined by equation (10) and the solution curve, at the end of integration: the correct scaling is then obtained either by using this value of B in equations (11) and (12), or appropriately adjusting the value of ξ_0. Thus we reduce the problem to an integral of a second-order differential equation (13) with the one-point boundary condition:

$$\xi=\xi_0: \qquad f=0; \qquad f^{n-1}\frac{\mathrm{d}f}{\mathrm{d}\xi}\to -\frac{nm+n+1}{(2m+1)C}\xi_0. \tag{17}$$

When $0<n<1$ all solutions which meet the ξ axis do so tangentially. We may also use the same argument as for $n>1$ to reduce the problem to a one-point boundary condition problem with the same boundary conditions (17) as before.

These integrals are most easily evaluated in terms of the function $y=f^n$ by standard numerical methods. Figure 1 shows profiles for the typical cases: $n=0{\cdot}5$, $n=2{\cdot}5$ (electron thermal conduction) and $n=5{\cdot}5$ (a typical value for radiative transport).

† The two conditions are not independent as is shown by integrating (13). Thus only one can be used as a boundary condition.

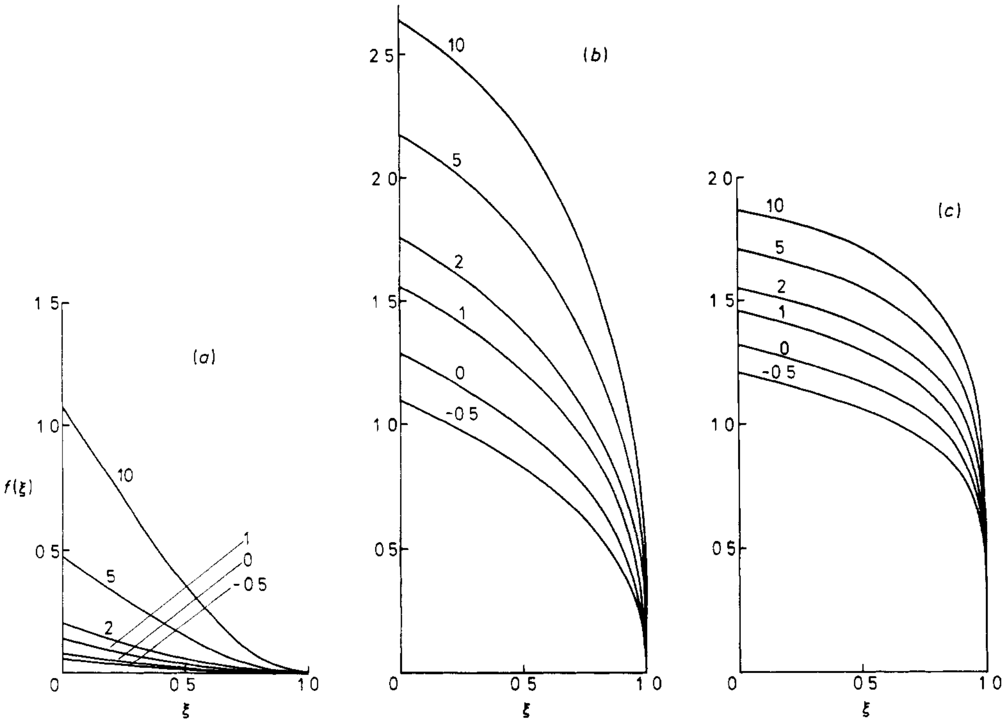

Figure 1. The characteristic profile function for typical values of m (−0·5, 0, 1·0, 2·0, 5·0 and 10·0) and: (a) $n = 0{\cdot}5$, (b) $n = 2{\cdot}5$, (c) $n = 5{\cdot}5$. The value of $C = (n+2)/(2m+1)$ and $\xi_0 = 1$.

5. Instantaneous heat deposition ($m = -1$)

As the case of instantaneous deposition is treated by Pattle (1959), Boyer (1962) and Žel'dovich and Raizer (1967), in detail, our purpose is solely to show that it is one of this class of self-similar solutions. As $m \to -1$, we see that the right-hand side of equation (8) diverges. However, if we consider the limiting procedure:

$$\lim_{m \to -1} S_0 \to Q(m+1) \qquad B \to b(m+1) \to 0 \tag{18}$$

we observe from equation (8) that the total energy has a constant value Q. In this case the differential equation (13) reduces to:

$$C\frac{\mathrm{d}}{\mathrm{d}\xi}\left(f^n\frac{\mathrm{d}f}{\mathrm{d}\xi}\right) = f + \xi\frac{\mathrm{d}f}{\mathrm{d}\xi} \tag{19}$$

which may be integrated directly. The boundary condition (16) or (17) used earlier is not applicable in this case, as it appears as the first integral of the differential equation and is therefore valid for all solutions of (13). Instead we may use the total energy form (10), which we rejected before, as in this case it is not an integral of the differential equation. The result of this calculation is given more generally in equations (27) and (28).

5.1. The case $m = -\frac{1}{2}$

We can treat the case $m = -\frac{1}{2}$, where $2m+1 \to 0$, in a similar manner to that of $m = -1$, by introducing the non-zero variable

$$c = (2m+1)C \tag{20}$$

in place of C.

5.2. The case $nm = 1$

When the product nm is unity, equation (13) has the simple analytic solution:

$$f(\xi)^n = \frac{n^2}{C}\xi_0(\xi_0 - \xi) \tag{21}$$

and

$$B = (n^2/C)^{(n+1)/n}\xi_0^{(n+2)/n}. \tag{22}$$

5.3. The case $n = 0$

When the thermal conductivity is constant the point ξ_0 tends to infinity, and the numerical integration technique described earlier is not suitable. We therefore present an analytical solution for this case. When m is integral or half-integral there is a well known solution in terms of the repeated integrals of the complementary error function (Carslaw and Jaeger 1959) and the similarity variable ξ is given by the Boltzmann transformation. This solution may be readily generalized to include arbitrary values of m by the use of the function $I_m(x)$ whose properties are discussed in the appendix. In terms of this function the solution for $f(\xi)$ is

$$f(\xi) = B' I_{2m+1}(\xi/\surd(2c)) \tag{23}$$

where $c = (2m+1)C$ and B' is arbitrary. From the properties of I_m we obtain:

$$B = \frac{1}{2^{2m}\Gamma(1+m)}\frac{B'}{\surd(2c)} \tag{24}$$

and hence

$$\epsilon(x, t) = 2^{2m+1}\Gamma(1+m)a^{-1/2}S_0 t^{m+\frac{1}{2}} I_{2m+1}(x/2\surd(at)). \tag{25}$$

6. Extension to other geometries

We may easily generalize the results to consider cylindrical and spherical geometries. Thus we consider in the cylindrical case a circular sector of 1 rad angle heated along its axis, and in the spherical case a cone of 1 sr solid angle heated at its apex. In this case we find that f is defined by

$$C\frac{1}{\xi^\nu}\frac{d}{d\xi}\left(\xi^\nu f^n \frac{df}{d\xi}\right) = f - \frac{nm+n+1}{2m+1-\nu}\xi\frac{df}{d\xi} \tag{13a}$$

where

$$\lim_{\xi\to 0}\left(\xi^\nu f^n \frac{df}{d\xi}\right) = -B$$

also

$$\int_0^\infty \xi^\nu f\, d\xi = \frac{2m+1-\nu}{(m+1)[n(\nu+1)+2]}BC. \tag{10a}$$

The functions ξ and $h(t)$ are:

$$\xi = x\left[\left(\frac{n(\nu+1)+2}{(2m+1-\nu)C}\right)^{n+1}\left(\frac{S_0}{B}\right)^n a t^{nm+n+1}\right]^{-1/[n(\nu+1)+2]} \tag{11a}$$

and

$$h(t) = \left[\left(\frac{(2m+1-\nu)C}{(n(\nu+1)+2}\right)^{\nu-1}\left(\frac{S_0}{B}\right)^2 a^{-(\nu+1)} t^{2m+1-\nu}\right]^{1/[n(\nu+1)+2]} \tag{12a}$$

where ν is the geometrical factor (0, planar; 1, cylindrical; 2, spherical). The boundary condition on f at ∞, namely $f\to 0$ as $\xi\to\infty$, is easily shown to reduce to (16).

We must, however, consider the behaviour of solutions in the neighbourhood of $\xi = 0$ in some detail. Thus expanding $f(\xi)$ near $\xi = 0$ in the power series:

$$f(\xi) = f_0 + f_1\xi + f_2\xi^2 \ldots \tag{26}$$

and substituting in (13a) we obtain:

$$\frac{\nu f_0^n f_1}{\xi} + (\nu+1)\left(\frac{nf_1}{f_0} + \frac{2f_2}{f_1}\right) f_0^n f_1 \ldots = f_0 + f_1\left(1 - \frac{mn+n+1}{2m+1-\nu}\right)\xi + \ldots. \tag{27}$$

Hence solutions to (13a) only exist if:

$$f_0 = 0 \qquad \text{or} \qquad f_1 = 0 \qquad \text{or} \qquad \nu = 0. \tag{28}$$

Clearly the first of these is inconsistent with the boundary conditions, and the third represents the planar case already discussed. The case $f_1 = 0$, corresponding to $B = 0$, only has a non-zero solution if $m = -1$, and therefore corresponds to the case of instantaneous energy deposition as discussed before.

The absence of solutions with a finite heat flux at the origin in the cylindrically or spherically symmetric cases is easily seen by considering the behaviour of the total heat flux at the origin $\kappa r^\nu\, d\epsilon/dr = S$, or equation (10$a$) which requires

$$\lim_{\xi\to 0}\frac{1}{n+1}f^{n+1} = -B\int \xi^{-\nu}\, d\xi \tag{29}$$

in order to remove the applied heat flux at the origin.

6.1. *Instantaneous energy deposition*

As we have shown, the only solution if $\nu \neq 0$, occurs if $m = -1$ when

$$f(\xi)^n = \frac{n}{2(\nu+1)(-C)}(\xi_0^2 - \xi^2) \tag{30}$$

where the position of the thermal front is given by

$$\xi_0^{[n(\nu+1)+2]/n}=\frac{b}{n(\nu+1)+2}\left(\frac{1}{n}\right)^{1/n}[2(\nu+1)(-C)]^{1+(1/n)}\frac{\Gamma(\frac{1}{2}\nu+\frac{3}{2}+n^{-1})}{\Gamma(1+n^{-1})\Gamma(\frac{1}{2}\nu+n^{-1})}. \tag{31}$$

This solution has been previously discussed by Pattle (1959) and Boyer (1962).

7. Surface temperature varying with time

Thus far we have considered the case where the flux at the surface $x=0$ is specified as a function of time. As mentioned earlier we can also find self-similar solutions when the surface temperature is specified as a simple power of time

$$\epsilon=\epsilon_0 t^l. \tag{32}$$

In this case we may carry out a similar analysis as before to derive the similarity form. However, the solution is most easily obtained directly from equations (11) and (12) or (11a) and (12a), by noting that in these cases the surface temperature varies as the $(2m+1-\nu)/[n(\nu+1)+2]$th power of time. Hence the solution when the surface temperature is specified is directly obtained from (11a) and (12a) by the substitution

$$l=(2m+1-\nu)/[n(\nu+1)+2] \tag{33}$$

and is

$$\epsilon=\frac{\epsilon_0}{B}t^l f\left(\frac{x}{[(a/lC)(\epsilon_0/B)t^{(ml+1)}]^{1/2}}\right) \tag{34}$$

where

$$B=f(0). \tag{35}$$

The function f is, of course, simply that given by (13a) with the appropriate substitution (33) for m in terms of l, and we have retained the arbitrary constants B and C.

We therefore conclude that unless $m=-1$, i.e. $l=-(\nu+1)/[n(\nu+1)+2]$, there are no solutions in the cylindrical or spherical cases.

We now give some simple examples of this substitution.

(1) $l=-(\nu+1)/[n(\nu+1)+2]$. In this case the total heat energy is a constant, $m=-1$, and the solution is given by (30) with:

$$\xi_0^2=\frac{2(\nu+1)(-C)B^n}{n}. \tag{31a}$$

This solution was previously given by Ames (1968).

(2) $nl=1$. In this case only planar solutions occur and $l=m$. The solution is given by (21) with

$$\xi_0^2=\frac{CB^n}{n^2} \tag{22a}$$

(3) $n=0$. In this case also only planar solutions need be considered when $2l=2m+1$.

Hence the solution is given by a comparison with (23):

$$f(\xi) = BI_{2l}(\xi/2\sqrt{(lC)}) \tag{23a}$$

where

$$B = 2^{2l}\Gamma(1+l) \tag{24a}$$

and $I_l(\xi)$ is the function described in the appendix. In the case that l is integral or half-integral $I_{2l}(\xi)$ is a repeated integral or the error function and the solution reduces to that discussed by Carslaw and Jaeger (1959).

8. Conclusions

We have shown that there exists a class of self-similar solutions to the one-dimensional non-linear heat conduction equation, when the thermal conductivity can be expressed as a simple power of the specific internal energy. These solutions may be applied to the case where the medium is initially very cold, and the applied flux (or surface temperature) varies on a simple power of time: the solutions being valid for values of the powers $n \geqslant 0$ and $m \geqslant -1$. We have also demonstrated how by a simple application of the scaling factors B and C, the complete solution may be easily evaluated.

These similarity solutions were originally developed to use as test problems for thermal conduction routines in general laser interaction codes. They have proved very suitable for this application, and it is believed that the solutions with increasing energy flux provide a more realistic test than the simpler instantaneous energy deposition solution used hitherto.

However, these solutions are of some interest in laser–plasma interaction studies. It is well known that during the early stages of laser irradiation of solid targets non-linear thermal conduction is the dominant process, whilst the thermal front is ahead of the head of the rarefaction wave. In the past this behaviour has been studied by means of the instantaneous energy deposition solution (Babuel-Peyrissac *et al* 1969, Caruso and Gratton 1969, Saltzmann 1973), which is suitable in applications with very short (of the order of picosecond) laser pulses. With the present interest in studies with pulses of order 100 ps, this approximation is no longer valid, and the present solutions are more appropriate. Under these conditions we are generally interested in the cases $n \geqslant 2$, so that the energy profile is relatively square. Hence the sound speed is approximately:

$$C_s = \gamma\bar{\epsilon}^p \tag{36}$$

where γ and p are constants and the mean specific internal energy

$$\bar{\epsilon} = \frac{1}{(m+1)\xi_0}\left[\left(\frac{(2m+1)C}{n+2}\right)^{n+1}\frac{S_0^2 t^{2m+1}}{B^n}\right]^{1/(n+2)}. \tag{37}$$

Hence we can calculate the time t_0 at which the rarefaction wave overtakes the thermal front:

$$\xi_0 g(t_0) = \int_0^{t_0} C_s \, \mathrm{d}t. \tag{38}$$

As this general result is easily obtained but rather complicated in form, we shall not present it here.

Appendix. The function $I_m(x)$

We seek to find the function $I_m(x)$ which satisfies the differential equation:

$$\frac{d^2 I_m}{dx^2} + 2x\frac{dI_m}{dx} - 2mI_m = 0 \tag{A.1}$$

subject to the boundary condition $I_m(x) \to 0$ as $x \to \infty$. There is a simple relationship amongst the solutions of this equation, namely if

$$I_{m+1} = \int_x^\infty I_m \, dx \qquad I_m = -\frac{dI_{m+1}}{dx} \tag{A.2}$$

and

$$I_{m-2} - 2xI_{m-1} - 2mI_m = 0 \tag{A.3}$$

then clearly I_m is a solution of (A.1).

Consider the series

$$I_m = \sum_{l=0} \frac{1}{2^{m-l}\Gamma(1+\frac{1}{2}(m-l))l!}(-x)^l. \tag{A.4}$$

It is easily seen that this function satisfies (A.1)–(A.3). To investigate the behaviour as $x \to \infty$ we note that

$$I_m = \frac{1}{2^m\Gamma(1+\frac{1}{2}m)}M(-\tfrac{1}{2}m, \tfrac{1}{2}, -x^2) - \frac{x}{2^{m-1}\Gamma((m+1)/2)}M(-(m-1)/2, \tfrac{3}{2}, -x^2) \tag{A.5}$$

$$= e^{-x^2}\left(\frac{1}{2^m\Gamma(1+\frac{1}{2}m)}M((m+1)/2, \tfrac{1}{2}, x^2) - \frac{x}{2^{m-1}\Gamma((m+1)/2)}M(1+\tfrac{1}{2}m, \tfrac{3}{2}, x^2)\right) \tag{A.6}$$

$$= \frac{e^{-x^2}}{\pi^{1/2}\Gamma(m+1)}\sum_l \Gamma\left(\frac{m+l+1}{2}\right)\frac{(-2x)^l}{l!} \tag{A.7}$$

where we have used Kummer's transformation for the confluent hypergeometric functions $M(a, b, x)$ (Slater 1960). We may now cast this sum in the form of a Mellin–Barnes integral:

$$I_m = \frac{e^{-x^2}}{(\pi)^{1/2}2\pi i\Gamma(m+1)}\int_{-\infty}^{\infty} \Gamma(-S)\Gamma((m+1+s)/2)(2x)^s \, ds \tag{A.8}$$

where the path of integration is chosen to include the poles at $s = 0, 1, 2, \ldots$ but exclude those at $s = -(m+1)/2, -(m+3)/2, \ldots$. Hence by deforming the path of integration to include only the poles $s = -(m+1+2\nu)/2$ we obtain the asymptotic form:

$$I_m \to \frac{2}{\sqrt{\pi}}\frac{e^{-x^2}}{\Gamma(m+1)(2x)^{m+1}}\sum_{\nu=0}\frac{\Gamma(2\nu+m+1)}{\nu!(2x)^{2\nu}}. \tag{A.9}$$

Hence $I_m(x) \to 0$ as $x \to \infty$, satisfying the boundary condition on I_m, and also on $f(\xi)$ (equation (14)).

These formulae represent a generalization of the well known results for the repeated integral of the error function. Thus if m is an integer it is easily seen that

$$I_m(x) = i^m \operatorname{erfc}(x) \qquad m \text{ integral.} \tag{A.10}$$

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